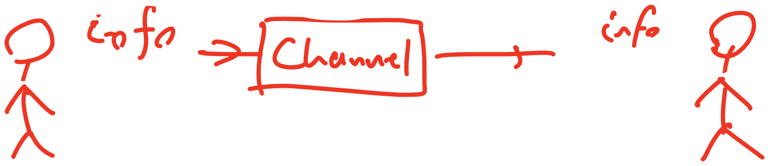
Lec 8: Quantum coding over quantum channels



Channel \ info	classical	Quantum
classical	✓	✓
quantum	✗	?

Classical info over quantum channel

$$\mathcal{N}: \mathcal{L}(H_A) \to \mathcal{L}(H_B)$$

$$[2^{nR}] \ni m \longrightarrow \boxed{\text{Enc}} \longrightarrow \boxed{\mathcal{N}^{\otimes n}} \longrightarrow \boxed{\text{Dec}} \longrightarrow \hat{m} \in [2^{nR}]$$

Enc: $[2^{nR}] \longrightarrow \mathcal{L}(H_A^{\otimes n})$ extends to a quantum ch.

$$\mathcal{L}(\mathbb{C}^{2^{nR}}) \longrightarrow \mathcal{L}(\mathcal{H}_A^{\otimes n}) \qquad |i\rangle\langle i| \longmapsto \text{Enc}(i)$$

Same is true for Dec

Now $\mathrm{Dec} \circ \mathcal{N}^{\otimes n} \circ \mathrm{Enc}$ is a channel from $\mathbb{C}^{2^{nR}} \longrightarrow \mathbb{C}^{2^{nR}}$ that preserveses computational basis $|i\rangle \longmapsto \approx |i\rangle$

Transmission of quantum info: Enc, Dec are such that $\mathrm{Dec} \circ \mathcal{N}^{\otimes n} \circ \mathrm{Enc}$ preserves all input states

More formally, given n uses of channel $\mathcal{N}$ a code with rate R is a pair $(\mathrm{Enc}, \mathrm{Dec})$:

$$\mathrm{Enc} : \mathcal{L}(\mathbb{C}^{2^{nR}}) \longrightarrow \mathcal{L}(H_A^{\otimes n})$$

$$\mathrm{Dec} : \mathcal{L}(H_B^{\otimes n}) \longrightarrow \mathcal{L}(\mathbb{C}^{2^{nR}})$$

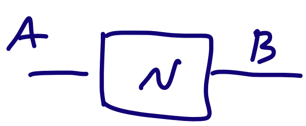
$$\forall\ |\varphi\rangle \in \mathbb{C}^{2^{nR}} \qquad \mathrm{Dec}(\mathcal{N}^{\otimes n}(\mathrm{Enc}(|\varphi\rangle\langle\varphi|))) \approx |\varphi\rangle\langle\varphi|$$

Error of code : $\sup_{|\varphi\rangle} \| \mathrm{Dec} \circ \mathcal{N}^{\otimes n} \circ \mathrm{Enc}(|\varphi\rangle\langle\varphi|) - |\varphi\rangle\langle\varphi| \|_1$

R is achievable for $\mathcal{N}$: $\exists$ seq of codes with rate R and vanishing error

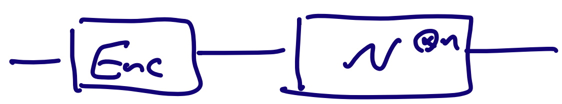
$Q(\mathcal{N}) =$ sup of all achievable rates

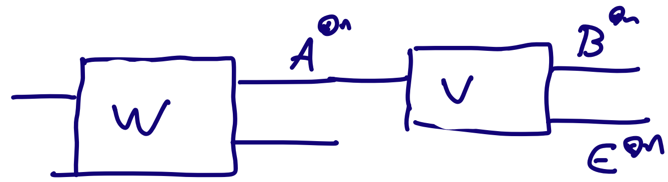
$Q(\mathcal{N}) \leq$ classical capacity



$\exists$ isometry $V : A \longrightarrow BE$

$$\mathcal{N}(\rho) = \mathrm{tr}_E(V \rho V^\dagger)$$





It makes sense to choose Enc as unitary

$M\,M'$ two reg with dim 2^{nQ}

$$|\varphi\rangle = \frac{1}{\sqrt{2^{nR}}} \sum_{i=1}^{2^{nR}} |i\rangle\,|i\rangle$$

$$\rho_{E^n R} = \left(1_{M'} \otimes N^c_{A\to E} \circ \mathrm{Enc}\right)\left(|\Phi\rangle\langle\Phi|\right)$$

$$\| \rho_{E^n R} - \rho_{E^n} \otimes \rho_R \|_1 \leq p \Longrightarrow \exists \text{ Dec with error } \leq \sqrt{p}$$