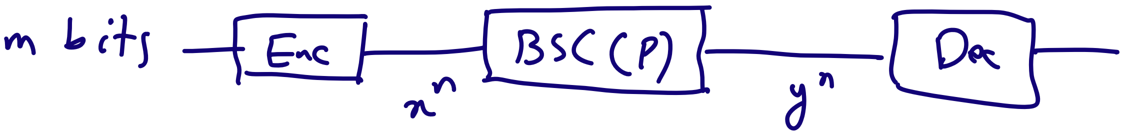
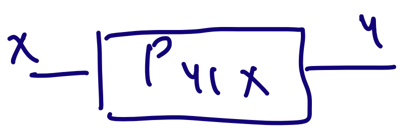
Lecture 7: Classical & quantum channel coding



We say rate R achievable if $\exists$ seq of codes with vanishing prob of error & rate $\to R$

max achievable rate $1 - h_2(P)$ Capacity

General Channel:



Capacity is max achievable rate

Thm: $\text{Capacity} = \max_{P_X} I(X:Y)$

$X, Y \sim P_X \times P_{Y|X}$

BSC(P): $\max_{P_X} I(X:Y) = 1 - h_2(P)$

$$I(X:Y) = H(Y) - H(Y|X) = H(Y) - \sum_x P(x) \underbrace{H(Y|X=x)}_{h_2(P)}$$

$$= H(Y) - h_2(P)$$

maximized when P_X is uniform

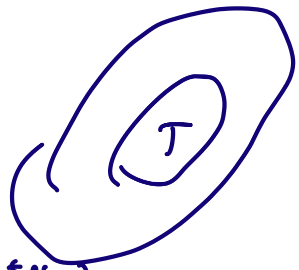
Fix P_X $\quad I(X:Y) = D(P_{XY} \| P_X \times P_Y)$

Look at hypothesis testing

$H_0 : (P_{XY})^{\otimes n}$

$H_1 : (P_X \times P_Y)^{\otimes n}$

$\mathcal{X}^n \times \mathcal{Y}^n$



Stein lemma $\quad \exists\ T \subseteq \mathcal{X}^n \times \mathcal{Y}^n$

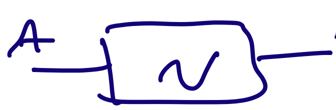
$P_{XY}^{\otimes n}(T) \geq 1 - \varepsilon$

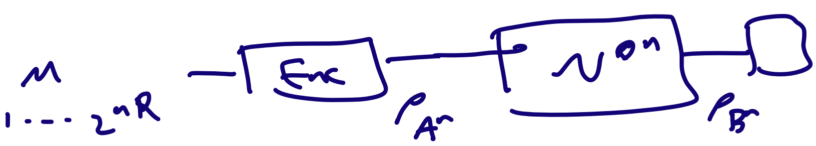
$(P_X \times P_Y)^{\otimes n}(T) \leq 2^{-nD(P_{XY} \| P_X \times P_Y) + o(n)}$

Choose $\quad x^n(1) \cdots x^n(2^{nR})$ iid. $P_X^{\otimes n}$

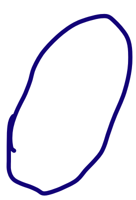
$$g(y^n) = \begin{cases} i & \text{if } \exists!\ i : (x^n(i), y^n) \in T \\ \# & \text{otherwise} \end{cases}$$

Quantum Channels

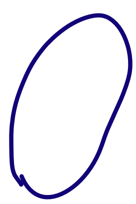




R achievable



density of $\mathbb{C}^n$



density of $\mathbb{C}^m$

$(p_1, \rho_1) \cdots (p_k, \rho_k)$ $\quad \rho_{XB} = \sum_{i=1}^{k} p_i |i\rangle\langle i| \otimes \mathcal{N}(\rho_i)$

$$C_H(\mathcal{N}) = \sup_{(p_1,\rho_1)\cdots(p_K,\rho_K)} I(X:B)_\rho$$

Observation: $\dfrac{C_H(\mathcal{N}\otimes\mathcal{N})}{2}$ is also achievable

Superadditivity: $\exists\ \mathcal{N} : \dfrac{C_H(\mathcal{N}\otimes\mathcal{N})}{2} > C_H(\mathcal{N})$

Thm Capacity $= \lim\limits_{r\to\infty} \dfrac{C_H(\mathcal{N}^{\otimes r})}{r}$