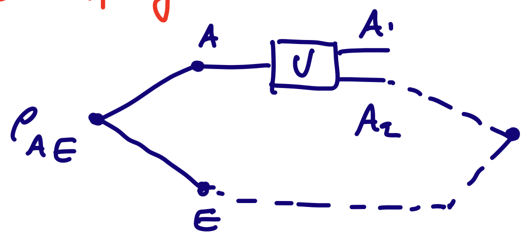
Lec 19:

Decoupling lemma

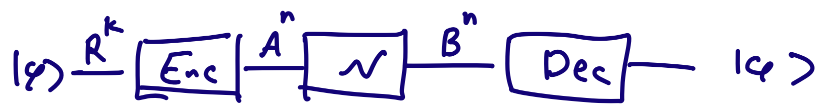


$$\rho^U_{A_2E} = \mathrm{tr}_{A_1}\left[(1_E \otimes U)\,\rho_{AE}\,(1_E \otimes U^\dagger)\right]$$

if U random & $|A_1|$ is large $\Longrightarrow \rho^U_{A_2E} \approx \frac{1}{|A_2|} \otimes \rho_E$

Thm $\mathop{\mathbb{E}}_{U\sim \mathrm{Haar}} \left\| \rho^U_{A_2E} - \frac{1}{|A_2|} \otimes \rho_E \right\|_1 \leq \frac{|A_2|}{|A_1|} \cdot \mathrm{tr}(\rho^2_{AE})$

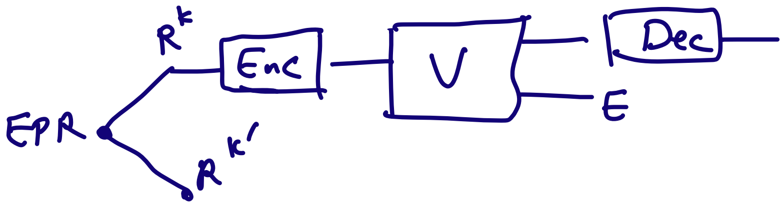
- quantum random extraction
- quantum error correction



k-qubit $\qquad$ n qubits

$(k<n)$

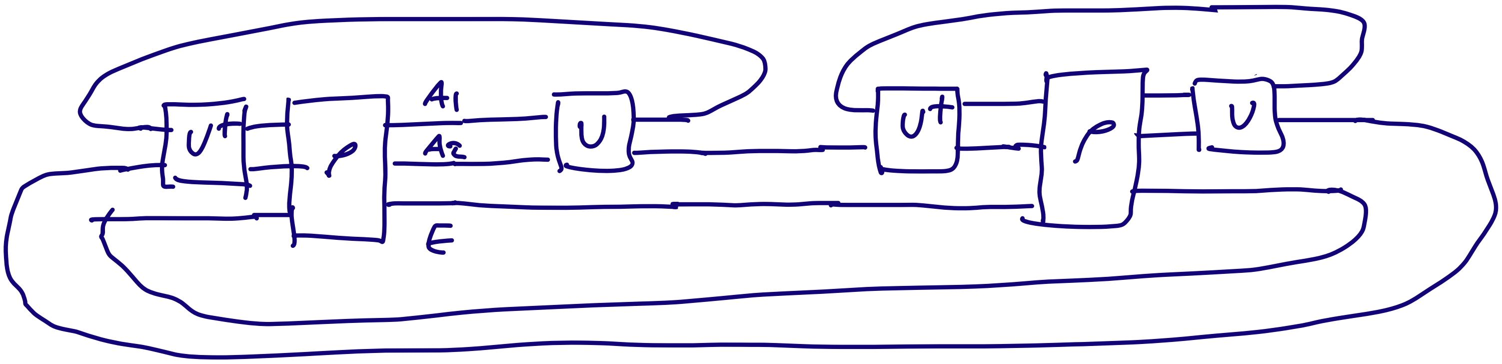
(stinespring extension)

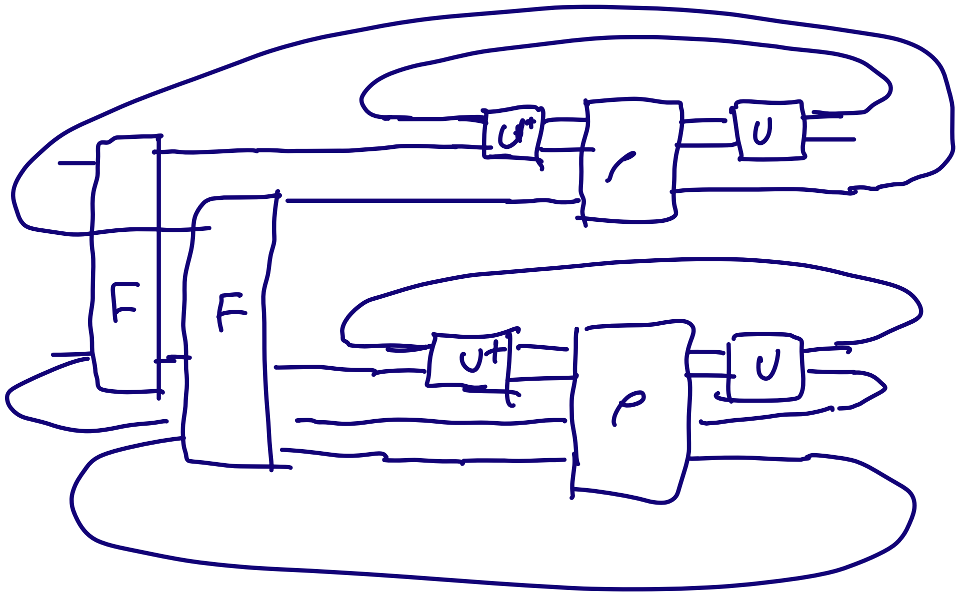


if $\rho_{R^{k'}R^kE} \approx \rho_{R^{k'}R^k} \otimes \rho_E \Longrightarrow \exists$ Dec s.t. $\mathrm{Dec} \circ \mathcal{N}^{\otimes n} \circ \mathrm{Enc} \simeq \mathrm{id}$

$$\mathop{\mathbb{E}}_U \left\| \rho^U_{A_2E} - \frac{1}{|A_2|} \otimes \rho_E \right\|_2^2$$

$$\leq \mathop{\mathbb{E}}_U \mathrm{tr}\left((\rho^U_{A_2E})^2 \right)$$





$$\mathrm{tr}\Big(\big(1 \otimes F_{A_2A_2'} \otimes F_{EE'}\big)\big((1\otimes U)\rho(1\otimes U)\big)^{\otimes 2}\Big)$$

$$\mathrm{tr}\Big(\Big((U^\dagger \otimes U^\dagger)\big(1_{A_1A_1} \otimes F_{A_2A_2}\big)(U \otimes U) \otimes F_{EE'}\Big)\rho^{\otimes 2}\Big)$$

$$Y = \mathbb{E}_{U\sim \mathrm{Haar}} {U^\dagger}^{\otimes 2} \, X \, U^{\otimes 2} = ?$$

$$[Y, U^{\otimes 2}] = 0 \implies Y = c_I 1 + c_F F$$

$$\mathrm{tr}\, X = \mathrm{tr}\, Y = c_I \,\mathrm{tr}(1) + c_F \,\mathrm{tr}(F) = c_I d^2 + c_F d$$

$$\mathrm{tr}(XF) = \mathrm{tr}(FY) = c_I d + c_F d^2$$

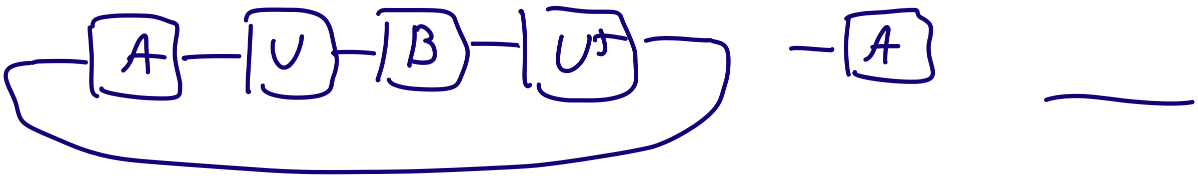
Tensorization:

$$\mathbb{E}_U \,\mathrm{tr}\big[(U A U^\dagger)\big]^2$$

$$= \mathbb{E}_U \,\mathrm{tr}\big[(UAU)^{\otimes 4}\big] = \mathrm{tr}\Big(\mathbb{E}_U (UAU)^{\otimes 2}\Big)$$

$$\mathbb{E}\ tr(AUBU^\dagger)^2 = \mathbb{E}\ tr(AUBU^\dagger \otimes AUBU^\dagger)$$

$$= \mathbb{E}\ tr\big((A\otimes A)(U\otimes U)(B\otimes B)(U^\dagger\otimes U^\dagger)\big)$$



Concentration of meas.

If $f: S(\mathbb{C}^d) \longrightarrow \mathbb{R}$ is 1-Lip $|f(x)-f(y)| \leq |x-y|$

then $\Pr\big[\,|f(|\varphi\rangle) - \mathbb{E} f(|\varphi\rangle)| \geq \varepsilon\big] \leq 3\exp(-\delta d \varepsilon^2)$

$\delta > 0$ absolute constant.

Lem: $|\varphi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ unif.

$\rho = tr_2\big[|\varphi\rangle\langle\varphi|\big]$

$\Pr\big[\,\|\rho - \frac{1}{n}\|_2 \geq O(\frac{1}{\sqrt{m}})\big] < 4^{-n}$