Lec 18:

Uniform dist. over $S(\mathbb{C}^d)$, Symmetric subspace $\mathrm{Sym}^n(\mathbb{C}^d)$

Invariant under any unitary

$|\varphi\rangle$: $\sigma|\varphi\rangle = |\varphi\rangle \ \forall \sigma \in S_n$

$\mathrm{Proj} := \frac{1}{n!}\sum_\sigma \sigma = \Pi_{n,d}$

ONB

Dim $\binom{n+d-1}{n}$

$$\mathbb{E}_{|\varphi\rangle} |\varphi\rangle\langle\varphi|^{\otimes n} \propto \Pi_{n,d}$$

Remark: (Rep. th.) $\rho: G \to \mathcal{L}(\mathbb{C}^d)$

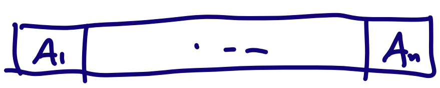
$g \mapsto A$ lin op. on $\mathbb{C}^d$

$\rho(gh) = \rho(g)\rho(h)$

$$V := \{|\varphi\rangle \in \mathbb{C}^d : \rho(g)|\varphi\rangle = |\varphi\rangle \ \forall g \in G\}$$

$$\text{Proj onto } V = \frac{1}{|G|}\sum_g \rho(g)$$

$$\mathrm{Sym}^n(\mathbb{C}^d) := \mathrm{span}\{|\varphi\rangle^{\otimes n} : |\varphi\rangle \in \mathbb{C}^d\}$$



$\rho^{\otimes n}$ ✓

ρ_{A^n}

Apply random permutation $\frac{1}{n!}\sum_\sigma \sigma \rho_{A^n} \sigma^\dagger = \sigma_{A^n}$

σ_{A^n} is sym. $\approx \sum \lambda_i \sigma_i^{\otimes n}$?

Quantum de Finetti's theorem

$|\Phi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d)$ $\quad 1 \leq k \leq n$

$\exists\ \tau_{A^k}$ density op on $(\mathbb{C}^d)^{\otimes k}$

convex combination of $(|\varphi\rangle\langle\varphi|)^{\otimes k}$

$$\| \mathrm{tr}_{A_{k+1}\cdots A_n}(|\Phi\rangle\langle\Phi|) - \tau \|_1 \leq \frac{4k(d-1)}{n+1}$$

– classical prob. – mixed states

Proof $\tau = \binom{n+d-1}{n} \mathop{\mathbb{E}}_{|\varphi\rangle} \langle\Phi| (|\varphi\rangle\langle\varphi|)^{\otimes n} |\Phi\rangle \, (|\varphi\rangle\langle\varphi|)^{\otimes k}$

Optimal cloner

$\Phi : |\varphi\rangle^{\otimes n} \longmapsto |\varphi\rangle^{\otimes m}$ $\quad m > n$

$$\alpha(\Phi) := \inf_{|\varphi\rangle} \langle\varphi|^{\otimes m} \Phi(\varphi^{\otimes n}) |\varphi\rangle^{\otimes m}$$

optimal value $:= \dfrac{\binom{n+d-1}{n}}{\binom{m+d-1}{m}}$

$$\alpha(\Phi) \leq \mathop{\mathbb{E}}_{\varphi} \langle\varphi|^{\otimes m} \Phi(\varphi^{\otimes n}) |\varphi\rangle^{\otimes m}$$

$$\leq \mathop{\mathbb{E}}_{\varphi} \langle\varphi|^{\otimes m} \Phi(\Pi_{n,d}) |\varphi\rangle^{\otimes m}$$

$$= \mathrm{tr}\left(\Phi(\Pi_{n,d}) \cdot \frac{\Pi_{m,d}}{\binom{m+d-1}{m}} \right)$$

$\leq$ ✓

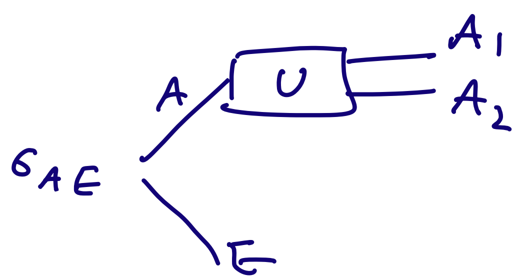
$$\Phi(X) := \frac{N_n}{N_m} \Pi_{m,d} (X \otimes 1) \Pi_{m,d} + \langle 1 - \Pi , X \rangle \sigma$$

$$\langle \varphi|^{\otimes n} \frac{N_n}{N_m} \Pi_{m,d} (\varphi^{\otimes n} \otimes 1) \Pi_{m,d} \; |\varphi\rangle^{\otimes m}$$

$$\frac{2}{d+1}$$

Haar meas.

Decoupling



$$\sigma^{U}_{A_2 E} := \mathrm{tr}_{A_1} \left(U_A \otimes 1_E \; \sigma_{AE} \; U_A \otimes 1_E \right)$$

$$\mathbb{E}_U \left\| \sigma^{U}_{A_2 E} - \frac{1_{A_2}}{|A_2|} \otimes \sigma_E \right\|_1 \leq \frac{|A_2| \, |E|}{|A_1|} \mathrm{tr}\left(\sigma^2_{AE} \right)$$

quantum capacity of quantum channel — quantum privacy ampl