Lec 16: Haar meas. & its applications

Stabilizer codes

- CSS code are stabilizer code
- Other direction is almost true
- Error are related though algebraic properties of Pauli group
- Quantum computing: applying Clifford gates is easy
  CNOT, H, S

Symmetries in quantum info

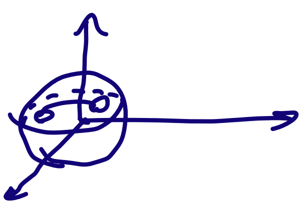
- uniform dist. over quantum states
  unitaries

Haar meas.

Random code / crypto

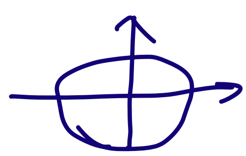
Uniform random state $\mathbb{C}^d$ $\quad S(\mathbb{C}^d) := \{|\varphi\rangle \in \mathbb{C}^d : \langle\varphi|\varphi\rangle = 1\}$

μ uniform (or Haar) if $\forall A \; \forall U \quad \mu(A) = \mu(UA)$



- Existence
- Uniqueness

$$|\tilde{\varphi}\rangle := \sum_{j=1}^{d} (x_j + i y_j)|j\rangle$$



$$|\varphi\rangle := \frac{|\tilde{\varphi}\rangle}{\|\tilde{\varphi}\|}$$

$$\mathop{\mathbb{E}}_{|\varphi\rangle\sim\mu} |\varphi\rangle\langle\varphi| = T = \sum \lambda_i |e_i\rangle\langle e_i|$$

$$\lambda_i = \langle e_i|T|e_i\rangle = \mathbb{E}\,|\langle\varphi|e_i\rangle|^2$$

$\underline{\mathbb{E}\, |\langle \varphi | e_i \rangle|^2} = \mathbb{E}\, |\langle \varphi | e_j \rangle|^2 = 1/n$

$\mathbb{E}\left(|\varphi\rangle\langle\varphi|\right)^{\otimes n} = T \quad (\mathbb{C}^d)^{\otimes n}$ Our goal is to find T:

σ permutation on n elements $\quad (\mathbb{C}^d)^{\otimes n} \longrightarrow (\mathbb{C}^d)^{\otimes n}$

$\qquad |\varphi_1\rangle \cdots |\varphi_n\rangle \qquad |\varphi_{\sigma^{-1}(1)}\rangle \cdots |\varphi_{\sigma^{-1}(n)}\rangle$

$\mathrm{Sym}^n(\mathbb{C}^d) := \{ |\varphi\rangle \in (\mathbb{C}^d)^{\otimes n} : \sigma|\varphi\rangle = |\varphi\rangle \}$

$|\varphi\rangle^{\otimes n} \in \mathrm{Sym}^n(\mathbb{C}^d)$

$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \in \mathrm{Sym}^2(\mathbb{C}^2)$

$\Pi = \frac{1}{n!} \sum_\sigma \sigma$ / Finding basis

$w : [d] \to \mathbb{Z}^{\geq 0}$

$\sum_a w(a) = n$

$|a\rangle := |a_1\rangle \cdots |a_n\rangle \qquad a_i \in [d]$

type of $|a\rangle$ is w if $\#i$ s.t. $a_i = a$ is $w(a)$

w/n is emperical dist. of sequence $(a_1 \cdots a_n)$

Example $|0100\rangle$

Fix w & Let T_w be all $|a\rangle$ s.t. type of $|a\rangle$ is w

$\Rightarrow |w) := \frac{1}{|T_w|^{1/2}} \sum_{|a\rangle \in} |a)$

dim of $\mathrm{Sym}^n(\mathbb{C}^d) = \binom{d+n-1}{n}$

Thm: If $[U_1^{\otimes n}, X] = 0 \ \forall X \iff X = \sum c_\sigma \sigma$

$$T U^{\otimes n} = \mathbb{E}\, |\varphi\rangle^{\otimes n} \langle \varphi|^{\otimes n} U^{\otimes n}$$

$$\Rightarrow T = \text{Proj onto } \mathrm{Sym}^n(\mathbb{C}^d) \Big/ \binom{d+n-1}{n}$$

As a corallary we obtain that

$$\mathrm{Sym}^n(\mathbb{C}^d) = \mathrm{span}\{ |\varphi\rangle^{\otimes n} : |\varphi\rangle \in \mathbb{C}^d \}$$